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On the Cosmic and Local Mass Density of 'Invisible' Axions

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ABSTRACT

If the Universe is axion-dominated it may be possible to detect the axions which comprise the bulk of the mass density of the Universe. The feasibility of the proposed experiments depends crucially upon knowing the axion mass (or equivalently, the PQ symmetry breaking scale) and the local mass density of axions. In an axion-dominated Universe our galactic halo should be comprised primarily of axions. We calculate the local halo density to be at least $5\times10^{-25} gcm^{-3}$, and at most a factor of two larger. Unfortunately, it is not possible to pin down the axion mass, even to within an order of magnitude. In an axion-dominated Universe we place an upper limit to the axion mass of $2.5\times10^{-5}eV$. We give precise formulae for the axion mass in an axion-dominated Universe, and clearly point out all the uncertainties involved in determining the precise value of the mass.

Introduction

One of the most pressing issues in cosmology is the nature of the ubiquitous dark matter which prevades the Universe. In the past few years one of the most popular and attractive explanations has been that the dark matter is comprised of a cosmic sea of very weakly-interacting relic particles left over from an early, very hot epoch of the Universe. Candidate relics include massive neutrinos, photinos, superheavy monopoles and axions, just to mention a few! Of these many candidates, axions[1-3] are in many ways the most intriguing possibility. The energy density in axions corresponds to large-scale, coherent scalar field oscillations, set into motion by the initial misalignment of the field with the minimum of its potential[4,5]. These axions are very weakly-interacting (and in fact were originally dubbed 'invisible') and very cold (i.e., $v/c << 1$). Axions behave as cold dark matter[6] and as a result should be found in, and should be the dominant component of, the halos of spiral galaxies (including our own Galaxy).

Although cosmic axions were originally thought to be so weakly-interacting as to be invisible and undetectable, Sikivie[7] has recently pointed out that they might be detected by using a very strong, inhomogeneous magnetic field to convert cosmic axions to photons (taking advantage of the axion - photon-photon coupling through the axial anomaly). The feasibility of this experiment depends upon a number of factors including the mass density of halo material (presumed to be axions) in the solar neighborhood and the mass (or equivalently, the PQ symmetry breaking scale) of these axions. In this brief paper we comment on both of these issues.

Cosmic Density of Axions

Cosmic axions come into existence as coherent scalar field when the temperature of the Universe is about a GeV. These oscillations are set into motion by the initial misalignment of θ, the axion degree of freedom. Initially, when the Peccei-Quinn (PQ) symmetry is broken (temperature of order f_a) θ is left undertermined because the axion is massless. However, at low temperatures ($\lesssim$ order of a GeV) the axion develops a mass due to instanton effects. When the temperature of the Universe ($T \simeq T_1$) is such that the mass of the axion is about 3 times the expansion rate of the Universe, the coherent field oscillations commence[4]. Estimating the energy density in these oscillations depends upon many things including: f_a, the scale of PQ symmetry breaking; θ_1, the initial misalignment angle; and the finite-temperature behavior of the axion mass. A careful estimate of the energy density gives[4,5,11] (for a detailed discussion, see the Appendix)

$$(\Omega_a h^2/T_{2.7}^3)=1.0\times 10^{\pm 0.4} f(N\theta_1)(N/6)^{0.825}(f_a/10^{12}GeV)^{1.175}\theta_1^2\gamma^{-1}\Lambda_{200}^{-0.7}, \tag{1a}$$
$$(f_a\lesssim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3})$$
$$(\Omega_a h^2/T_{2.7}^3)=3.5\times 10^{6\pm 0.1} f(N\theta_1)(N/6)^{1/2}(f_a/10^{18}GeV)^{1.5}\theta_1^2\gamma^{-1}, \tag{1b}$$
$$(f_a\gtrsim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3})$$

where $\Omega_a=\rho_a/\rho_{crit}=1.88h^2\times 10^{-29}gcm^{-3}$ is the critical density, H=100h km Mpc^{-1} sec $^{-1}$ is the present value of the Hubble parameter, 2.7 $T_{2.7}K$ is the temperature of the microwave background radiation, Λ_{200} 200 MeV is the QCD scale factor, $f(N\theta_1)$ is a correction factor for anharmonic effects (for $N\theta_1\lesssim 1$, $f\simeq 1$; see Fig. A2 and the Appendix), N depends upon the PQ charges of the quarks (N=6 in the simplest models[8]), and γ is the ratio of the entropy per comoving volume now to that when $T \simeq T_1$. [Any entropy production since $T \simeq T_1$ dilutes the energy density in axions, which can only be calculated relative to that in photons. Entropy production could result due to the very out-of-equilibrium decay of a massive particle species, such as the gravitino[9].]

The initial misalignment angle θ_1 must be in the interval $[-\pi/N, \pi/N]$ as the axion potential is periodic with period $2\pi/N$. Although θ_1 is most likely to be of order unity, in inflationary models all values of θ_1 occur in some bubble or fluctuation region with finite probability[10,11]. If the Universe never underwent inflation, or if inflation occured before PQ symmetry breaking, then it is possible to accurately estimate θ_1. In that case, at the onset of coherent axion oscillations, θ_1 is correlated on the scale of the horizon, but is uncorrelated on larger scales. All values of θ_1 in the interval $[-\pi/N, \pi/N]$ are equally probable, and so the RMS value of θ_1 (which is what is relevant for computing the mean energy density of axions in the Universe) is just $(\pi/N)/\sqrt{3}$. Of course, in the case of inflation the RMS value of θ_1, averaged over *all* bubbles, is also $(\pi/N)/\sqrt{3}$. But since we live in a particular bubble, that fact is of no relevance (see Fig. 1).

The zero-temperature mass of the axion and the PQ symmetry breaking scale are related by

$$m_a = \frac{\sqrt{z}}{1+z}\frac{f_\pi m_\pi}{f_a} N = 0.48 N f_\pi m_\pi / f_a, \tag{2a}$$

$$= 3.7(N/6) \times 10^{-5} eV (f_a/10^{12} GeV)^{-1}, \tag{2b}$$

where $z = m_u/m_d \simeq 0.55$.

Bringing together Eqns(1,2) we can solve for the predicted axion mass or symmetry breaking scale

$$(f_a/10^{12}GeV)=1.0\times10^{\pm0.35}(N/6)^{-0.7}(\Omega_a h^2\gamma/fT_{2.7}^{3})^{.85}\theta_1^{-1.7}\Lambda_{200}^{0.6}, \qquad (3a)$$

$$(m_a/10^{-5}eV)=3.7\times10^{\pm0.35}(N/6)^{1.7}\theta_1^{1.7}(\Omega_a h^2\gamma/fT_{2.7}^{3})^{-0.85}\Lambda_{200}^{-0.6}, \qquad (3b)$$

$$(f_a\lesssim1.6\times10^{18\pm1.7}GeV(N/6)\Lambda_{200}^{-2.3})$$

$$(f_a/10^{18}GeV)=4.3\times10^{-5\pm0.07}(N/6)^{-1/3}(\Omega_a h^2\gamma/fT_{2.7}^{3})^{2/3}\theta_1^{-4/3}, \qquad (4a)$$

$$(m_a/10^{-11}eV)=8.6\times10^{4\pm0.07}(N/6)^{4/3}(\Omega_a h^2\gamma/fT_{2.7}^{3})^{-2/3}\theta_1^{4/3}, \qquad (4b)$$

$$(f_a\gtrsim1.6\times10^{18\pm1.7}GeV(N/6)\Lambda_{200}^{-2.3})$$

Even taking $\Omega_a=1$, and ignoring the dependence of m_a on the initial misalignment angle there is a great deal of leeway; allowing the following uncertainties: $0.4\leq h\leq1$, $1\leq T_{2.7}\leq1.1$, $1/2\leq\Lambda_{200}\leq2$ and the intrinsic uncertainty in computing the axion energy density (see Appendix), results in a factor of about 8 uncertainty (either way) in the predicted axion mass. Eqn(3) does not take into account any systematic uncertainties, for example, in calculating the finite-temperature axion mass[12], which is crucial for determining T_1, or axion damping mechanisms which have been recently discussed (although the damping expected seems likely to be very small)[13].

In sum, it is probably fair to say that one cannot predict the axion mass to better than an order of magnitude. To this I might add that one can however place an ***upper limit*** to the axion mass in an $\Omega_a=1$, axion-dominated Universe, by taking $\Lambda_{200}=1/2$, $N\theta_1=\pi$, $\gamma=1$, $h=0.4$, $T_{2.7}=1.1$, and $\Omega_a=1$

$$m_a\lesssim1.1\times10^{-4\pm0.35}\ eV \qquad (4)$$

Local Mass Density of Axions

If axions are the dark matter, then they should provide the halo material in our Galaxy and other spiral galaxies[14]. Predicated upon this assumption Sikivie[7] has proposed an axion detection scheme which might be capable of detecting the local reservoir of axions. The feasibility of his (and other) detection scheme(s) depends upon the local mass density of axions which should just be the local halo mass density. As an estimate Sikivie takes

$$\rho_{halo} = 10^{-24} g cm^{-3} \tag{5}$$

In this section I will derive an estimate for ρ_{halo} based upon Bahcall's models[15] of the Galaxy which is about a factor of 2 smaller, and I believe more realistic.

The Galaxy is thought to consist of three components: the disk, the spherical bulge, and the extended halo (see Fig. 2). The mass density is given by

$$\rho_{TOT} = \rho_{disk} + \rho_{bulge} + \rho_{halo} \tag{6}$$

The halo is believed to be well represented by an isothermal sphere model

$$\rho_{halo}(r) = \rho_0/(r^2 + a^2), \tag{7}$$

where a is the core radius of the isothermal sphere. [Such models are believed to describe self-gravitating systems of non-interacting particles; in particular, they predict the flat rotation curves, i.e., $v_{rot} \simeq constant$, that are observed in virtually all spiral galaxies.]

Kepler's third law implies that the orbital velocity of a star in a circular orbit is given by

$$rv_{rot}^2 = GM_{halo}(r) + GM_{bulge}(r) + GM_{eq}(r), \tag{8}$$

where $M_{halo}(r)$ is the halo mass interior to the orbital radius r, $M_{bulge}(r)$ is the bulge mass interior to r, and $M_{eq}(r)$ is the equivalent central mass which is needed to account for the gravitational effect of the disk.

For $r >> R$ the contribution of the bulge and of the disk to the rhs of Eqn(8) is negligible, while for $r < R$, rv_{rot^2} is almost totally accounted for by the disk and bulge components. Here $R \simeq 9kpc$ is the distance from our position to the center of the Galaxy. For reference

$1\ pc{=}3.09\times 10^{18} cm$

$1\ M_\odot \simeq 1.99\times 10^{33} g$

$1\ M_\odot pc^{-3} \simeq 6.7\times 10^{-23} gcm^{-3}$

In terms of ρ_0 and a, $M_{halo}(r)$ is given by

$$\begin{aligned} M_{halo}(r) &= 4\pi\int_0^r \rho_{halo}(r) r^2 dr, \\ &= 4\pi\rho_0 r(a/r)\int_0^{r/a} x^2 dx/(1+x^2), \\ &\equiv 4\pi\rho_0 r J(r/a). \end{aligned} \tag{9}$$

The integral J is tabulated in Table 1 for $r/a{=}0.1$, 0.3, 1.0, 3.0, 10.0, 30.0, and 100. For $r/a{<<}1$, $J{\simeq}(r/a)^2/3$, and for $r/a{>>}1$, $J{\simeq}1$. Using the fact that $rv_{rot}^2 \simeq GM_{halo}(r)$ for $r{>>}R$ and $v_{rot}(r{>>}R){\simeq}220\ km\,\sec^{-1}$, we can solve for ρ_0 (for a discussion of the rotation curve of the Galaxy see ref. 17)

$$\rho_0 = 5.8\times 10^{20} g\ cm^{-1}, \tag{10a}$$

$$\rho_{halo}(R) = 7.5\times 10^{-25} g\ cm^{-3}/[1+(a/R)^2], \tag{10b}$$

$$M_{halo}(R) = 1.0\times 10^{11} M_\odot J(R/a), \tag{10c}$$

$$[GM_{halo}(r)/r]^{1/2} = 220 km\,\sec^{-1} J^{1/2}(r/a), \tag{10d}$$

$$\begin{aligned} \sigma_{halo}(R) &\equiv \int_{-\infty}^{\infty} \rho_{halo}[(R^2+z^2)^{1/2}]dz = \pi R\rho_{halo}(R)[1+(a/R)^2]^{1/2}, \\ &= 313 M_\odot pc^{-2}/[1+(a/R)^2]^{1/2}, \end{aligned} \tag{10e}$$

$$\sigma_{halo}(R, few\ kpc) = 55 M_\odot pc^{-2}/[1+(a/R)^2]. \tag{10f}$$

Here $\sigma_{halo}(R)$ is the total column density of halo material at our position and $\sigma_{halo}(R,\ few\ kpc)$ is the column density of halo material within a few kpc of the plane of the Galaxy at our position. Note that based upon $v_{rot}(r{>>}R)$ alone, the local halo density can be at most $7.5\times 10^{-25} g\ cm^{-3}$.

The orbital velocity at our radius is about 240 km $\sec^{-1}$. The Galaxy models of Bahcall and his collaborators[15] indicate that about half the orbital velocity squared at our position is accounted for by the gravitational effects of the bulge and disk components. From Eqn(10d) and Table 1 this indicates that R/a must be about 2, implying that

$$\rho_{halo}(R)=5\times 10^{-25} g\ cm^{-3}, \tag{11a}$$

$$\sigma_{halo}(R,\ few\ kpc)=40 M_{\odot} pc^{-2}. \tag{11b}$$

Bahcall and his collaborators have constructed detailed, three-component models of our Galaxy. Their models[15] indicate that

$$\rho_{halo}(R)=6\times 10^{-25} g\ cm^{-3}, \tag{11c}$$

a local density which is very consistent with my estimate. In addition, Bahcall[16] has constructed detailed models of the distribution of matter in the vicinity of the sun. He uses the observed motions of stars perpendicular to the galactic disk to determine the total amount of local matter, and obtains

$$\rho_{TOT}(R)=0.2 M_{\odot} pc^{-3}=1.2\times 10^{-23} g\ cm^{-3}, \tag{12a}$$

$$\sigma_{TOT}(R,\ few\ kpc)=70 M_{\odot} pc^{-2}, \tag{12b}$$

Some of the quantities that he calculates in his models can be determined by a direct inventory of material in the solar neighborhood. In particular

$$\rho_{seen}(R)=0.095 M_{\odot} pc^{-3}=6\times 10^{-24} g\ cm^{-3}, \tag{13a}$$

$$\sigma_{seen}(R)=30 M_{\odot} pc^{-2}, \tag{13b}$$

where the 'seen' component includes all the material that has been detected in one way or another - stars, gas, dust, etc. Based upon his model of the solar neighborhood and the local inventory, Bahcall (as well as Oort[18] earlier) conclude that there are equal amounts of seen and unseen material in the solar neighborhood.

Could this unseen material be halo material? Bahcall (and I believe any reasonable person would) concludes NO. To see how implausible this hypothesis is, assume that the local halo mass density were this large and that the halo density interior to R is constant (i.e., $a>>R$)--which is a very conservative assumption. We would then find that due to the halo material alone

$$[GM_{halo}(R)/R]^{1/2}=360 km\ \mathrm{sec}^{-1}, \tag{14a}$$

$$\sigma_{halo}(R,\ few\ kpc)=450 M_{\odot} pc^{-2}, \tag{14b}$$

$$M_{halo}=2.7\times10^{11} M_{\odot}, \tag{14c}$$

which is clearly in conflict with the observational data. [Bahcall has not used his models of the local neighborhood to place an upper limit on σ (R, *few kpc*); however, it seems likely that such a large column of material would have a big effect on the motions of nearby stars perpendicular to the Galactic plane.] In addition, such a local density of halo material would result in:

$$v_{rot}(r>>R,\ a)=620(a/R) km\ \mathrm{sec}^{-1}, \tag{15}$$

(based on the isothermal model) - also clearly absurd. Bahcall[16] concludes that the unseen material must be in the form of a dark, disk component. Clearly this cannot be axions as they have no way to dissipate their gravitational energy, which they would have to do in order to settle into the disk[19].

All of these estimates for the halo density are predicated on the assumption that the halo is well described by an isothermal sphere. Our knowledge of the ellipticity of the halo is poor; however, the few observations[20] which bear on this question seem to indicate that the ratio of minor/major axes is $\gtrsim 0.8$. One might have expected that the presence of the disk would tend to flatten the halo. Numerical simulations done by Barnes[21] indicate that this is likely to be a small effect, perhaps causing a spherical halo to be compressed to an elliptical halo with minor/major axis ratio of 0.8-0.9. If this were the case for the Galaxy, then the halo density in the Galactic plane *might* be 20%-40% larger than my estimates.

Based upon my simple analysis of the Galactic rotation curve and Bahcall's detailed models of the Galaxy, one would conclude that the local mass density of halo material must be at least

$$\rho_{halo}(R)=5\times10^{-25} g\ cm^{-3}, \tag{16}$$

in order to support the observed rotational velocities at $r>>R$. If a/R is $<<1$, if v_{rot} is significantly larger than 220 km sec^{-1}, or if the halo is highly non-spherical, then the local density could be a factor of 2 or so higher. My estimate is about a factor of 2 smaller than Sikivie's estimate[7].

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Table 1 - Numerical Evaluation of

$$J \equiv (a/r)\int_0^{r/a} x^2 dx/(1+x^2)$$

r/a	J	J
0.1	0.00331	0.0575
0.3	0.0285	0.169
1.0	0.215	0.464
3.0	0.584	0.764
10.0	0.853	0.924
30.0	0.949	0.974
100	0.984	0.992

Appendix

In this Appendix, I briefly outline how the formulae for the present axion density, Eqns(1a, b), are obtained. While this is basically a review of refs. 4, 5, and 11, I will pay particular attention to the uncertainties in the calculation. This Appendix is meant to subsume all the above references; in the process I have corrected errors and have been as accurate as possible.

The equation of motion for the axion degree of freedom θ is

$$\ddot{\theta} + 3H\dot{\theta} + V'(\theta) = 0 \tag{A1}$$

where for small θ, $V'(\theta) \simeq m_a^2\theta$--this is the usual assumption which is made, and for the moment I will take V' to be $m_a^2\theta$. At early times, when the axion mass is much less than 3H--corresponding to temperatures much greater than Λ_{QCD}, the solution to Eqn(A1) is

$$\theta \simeq const \equiv \theta_1.$$

When the axion mass is much greater than 3H, θ oscillates with angular frequency $m_1(T)$, and the axion energy density,

$$\rho_a \equiv \frac{1}{2} f_a^2 (m_a^2\theta^2 + \dot{\theta}^2),$$

evolves as

$$\rho_a R^3 / m_a = const. \tag{A2}$$

[I refer the reader to refs. 4, 5, and 22 for further details about the solution to Eqn(A1).]

It is also usually assumed that Eqn(A2) is valid for $T \leq T_1$, where T_1 is defined by

$$m_a(T_1) \equiv 3H(T_1).$$

Making this assumption and taking θ to be equal to θ_1 at $T = T_1$, it then follows that

$$\rho_a = \frac{1}{2} m_a(T_1) m_a \theta_1^2 f_a^2 (R_1/R)^3. \tag{A3}$$

where R_1 is the value of the cosmic scale factor when $T = T_1$.

I have integrated Eqn(A1) numerically, assuming that around $T = T_1$ the axion mass varies as T^{-m}. I find that Eqns(A2, 3) are not exactly correct, rather, for $T << T_1$,

$$\rho_a = cf \times [\frac{1}{2} m_a(T_1) m_a \theta_1^2 f_a^2 (R_1/R)^3]$$

where cf is a correction factor which depends upon m and is plotted in Fig. A1. My numerical results for cf are well fit by

$$cf = 0.44 + 0.25m.$$

I should emphasize that only the behavior of the axion mass for $T \simeq T_1$ is crucial for determining the correction factor cf. For $T << T_1$, corresponding to $m_a >> 3H$, Eqn(A2) is a very good approximation.

In order to find the average energy density in axions one would simply replace θ_1^2 by $<\theta_1^2>$. Assuming that all values of θ_1 are equally probable, $<\theta_1^2> = (\pi/N)^2/3 = \theta_{1RMS}^2$.

Let's return to the approximation $V' \simeq m_a^2 \theta$. This approximation is only valid for small θ; however, to find the average energy density we must average over all values of θ. Note, $N\theta_{1RMS} = \pi/\sqrt{3} > 1$. One might expect V to be α 1-cos($N\theta$), in which case $V'(\theta)\ \alpha\ \sin(N\theta)$ (see ref. 12). I have integrated Eqn(A1) using

$$V'(\theta)=m_a(T)^2\sin(N\theta)/N.$$

Due to anharmonic effects the energy density obtained using this form for V' is larger than that obtained using $V'\simeq m_a^2\theta$, by a θ-dependent factor $f(N\theta)$, which is plotted in Fig. A2. The primary effect here is that for $\theta>1$, $\sin\theta$ is flatter than θ and so the axion oscillations commence later. Taking into account these anharmonic effects the average energy density is obtained by using a value of θ_1^2 in Eqn(A3) equal to

$$<\theta_{eff}^2>\equiv\frac{1}{\pi}\int_0^\pi u^2 f(u)du/N^2.$$

For $m\simeq 0$–8, $<\theta_{eff}^2>\simeq(1.9-2.4)\theta_{1RMS}^2\simeq(\pi/N)^2/(1.2-1.6)$.

Assuming that the expansion has been adiabatic since $T=T_1$, the entropy per comoving volume, $S\ \alpha\ R^3 s$, remains constant, so that

$$(R_1/R)^3=(\frac{45}{2\pi^2})\frac{s}{g_*(T_1)T_1^3}. \qquad \text{(A4a)}$$

If the entropy per comoving has increased since that epoch, then

$$(R_1/R)^3=(\frac{45}{2\pi^2})\frac{s/\gamma}{g_*(T_1)T_1^3}. \qquad \text{(A4b)}$$

where $\gamma=S(today)/S(T=T_1)$ and s is the entropy density: $s=(2\pi^2/45)g_*T^3$. During the epoch when the axion field begins to oscillate the Universe is radiation-dominated and

$$H=1.66g_*(T)^{1/2}T^2/m_{pl}. \qquad \text{(A5)}$$

As usual $g_*(T)$ counts the total number of effective relativistic degrees of freedom:

$$g_*=\sum_{Bose}g_B+(7/8)\sum_{Fermi}g_F.$$

For the temperatures of interest $g_*(T)$ and the (known) relativistic species are

$g_*(T>5GeV) \geq 86.25$ $u, d, c, s, b; e, \mu, \tau; 3\nu\overline{\nu}; 8G, \gamma$

$g_*(T>2GeV) \geq 75.75$ $u, d, c, s; e, \mu, \tau; 3\nu\overline{\nu}; 8G, \gamma$

$g_*(T>\Lambda_{QCD}) \geq 61.75$ $u, d, s; e, \mu; 3\nu\overline{\nu}; 8G, \gamma$

$g_*(T>100MeV) \geq 17.25$ $\pi^{\pm}, \pi^0; e, \mu; 3\nu\overline{\nu}; \gamma$

$g_*(T>1MeV) \geq 10.75$ $e; 3\nu\overline{\nu}; \gamma$

Using Eqns(A4,5) the axion energy density to entropy density ratio can be written as

$$\rho_a/s = 5.7 g_*^{-1/2}(T_1) m_a f_a^2 \theta_1^2 / T_1 m_{pl}. \tag{A6}$$

The present axion energy density is then obtained by multiplying this by the present entropy density

$$s_{TODAY} \simeq 1.7 T^3 \simeq 7.04 n_\gamma \simeq 2809 T_{2.7}^3 \ cm^{-3}$$

In order to evaluate this expression for the axion energy density, T_1 must be determined. T_1 depends upon the finite temperature behavior of the axion mass. Recall that the axion mass arises due to instanton effects. Gross, Pisarski, and Yaffe[12] have calculated these effects using the dilute instanton gas approximation. In the high temperature limit, $T>>\Lambda_{QCD}$, the axion mass is given by an integral over instantons of all sizes (cf., Eqn(6.15) in ref. 12):

$$m_a^2(T) = \frac{N^2 \Lambda^4}{f_a^2} \frac{m_1 \cdots m_{N_f}}{\Lambda^{N_f}} (\Lambda/T)^{7+N_f/3} I, \tag{A7}$$

$$I \equiv 0.130078 (\frac{33-2N_f}{6})^6 \xi^{N_f-1} \int_0^\infty v^{6+N_f/3} [\ln(T/v\Lambda)]^{6-a} \times \ \exp[f(v)] dv, \tag{A8}$$

$$f(v) \equiv -\pi^2 v^2 (2+N_f/3) + (3/2-N_f/6)[\ln(1+\pi^2 v^2/3) - 12\alpha(1+\delta/\pi^{3/2} v^{3/2})^{-8}], \tag{A9}$$

where $\xi = 1.33876$, $\alpha = 0.01289764$, $\delta = 0.15858$, $a = (153-19N_f)/(33-2N_f)$, $\Lambda \equiv \Lambda_{QCD} = \Lambda_{200} 200 MeV$, and N_f is the number of light quark flavors, i.e., with mass $< T$. Since the temperatures of interest span the range from a few 100

MeV to many GeV, it is not clear what number should be chosen for N_f -- 2, 3, 4, or 5. Eqn(A7) has been numerically evaluated for N_f=1 - 6, and can be written in the following form:

$$m_a(T)/m_a = a\Lambda_{200}^{b}(\Lambda/T)^{c}[1-\ln(\Lambda/T)]^{d}, \tag{A10}$$

where

		a	b	c	d
$N_f =$	1	0.277	3/2	3.67	0.84
	2	0.0349	1	3.83	1.02
	3	0.0256	1/2	4.0	1.22
	4	0.0421	0	4.17	1.46
	5	0.118	-1/2	4.33	1.74
	6	0.974	-1	4.5	2.07

The $[ln(T/v\Lambda)]^{6-a}$ factor has been taken out of the integrand and evaluated at the value of v where most of the contribution to the integral I arises-- $v^{-1}\simeq e$=2.7182818.... The error made in doing this is typically less than 10%. The finite temperature axion mass calculated from Eqn(A10) is shown in Fig. A3 for N_f=2, 3, 4, 5. That $\ln[eT/\Lambda]$ vanishes for $T=\Lambda/e$ and then becomes negative is indicative of the fact that the dilute instanton gas approximation is a high-temperature approximation which breaks down at low temperatures. The power law forms

$$m_a(T)/m_a = 0.3(\Lambda/T)^{3.8},\ 0.02(\Lambda/T)^{3.6}$$

span the spread of the calculated axion masses for N_f=2, 3, 4, and 5. Based upon that, I will take the finite temperature axion mass to be

$$m_a(T)/m_a = 7.7\times 10^{-2\pm 0.6}(\Lambda/T)^{3.7\pm 0.1}. \qquad \text{(A11)}$$

Using this formula it is straightforward to solve for T_1:

$$T_1 = 1.2\times 10^{\pm 0.15}GeV(N/6)^{0.175\pm 0.003}(f_a/10^{12}GeV)^{-0.175\pm 0.003}\Lambda_{200}^{0.7}$$
$$(f_a \lesssim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3}) \qquad \text{(A12a)}$$
$$T_1 = 1.6\times 10^{-1\pm 0.03}GeV(N/6)^{1/2}(f_a/10^{18}GeV)^{-1/2},$$
$$(f_a \gtrsim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3}) \qquad \text{(A12b)}$$

where I have propagated the uncertainties in all quantities--$g_*(T_1)$, $m_a(T)$, etc., and the '$\pm$ factors' in the exponents are meant to be an estimate of the known uncertainty in the quantity calculated. For $f_a \gtrsim 1.6\times 10^{18}GeV(N/6)\Lambda_{200}^{-2.3}$, T_1 occurs at a value less than that where the extrapolated power law exceeds the zero temperature mass (in which case the zero temperature mass has been used).

Bringing everything together we obtain

$$(\Omega_a h^2/T_{2.7}^3) = 1.0\times 10^{\pm 0.4}f(N\theta_1)(N/6)^{0.825\pm 0.003}(f_a/10^{12}GeV)^{1.175\pm 0.003}\theta_1^2\Lambda_{200}^{-0.7}\gamma^{-1},$$
$$(f_a \lesssim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3}) \qquad \text{(A13a)}$$
$$(\Omega_a h^2/T_{2.7}^3) = 3.5\times 10^{6\pm 0.1}f(N\theta_1)(N/6)^{1/2}(f_a/10^{18}GeV)^{3/2}\theta_1^2\gamma^{-1},$$
$$(f_a \gtrsim 1.6\times 10^{18\pm 1.7}GeV(N/6)\Lambda_{200}^{-2.3}) \qquad \text{(A13b)}$$

where the estimated uncertainties, a factor of about 3 for $f_a \lesssim 1.6\times 10^{18}GeV$ and a factor of about 1.3 for $f_a \gtrsim 1.6\times 10^{18}GeV$, have been obtained by merely propagating the uncertainties discussed above. In addition, the correction factor to Eqn(A3) discussed above has also been included--a factor of 1.37 for Eqn(A13a) and a factor of 0.44 for Eqn(A13b). The factor $f(N\theta_1)$ is the correction factor for anharmonic effects (see Fig. A2). I caution to add that there may be other systematic effects which modify Eqn(A13), such as additional particle species (which increase g_*), errors associated with the dilute instanton gas approximation, the effect of the higher momentum modes on the evolution of the zero momentum mode considered here, additional anharmonic effects[22] and the possible effect of the chiral symmetry breaking transition on the evolution of $m_a(T)$ (refs. 13 and 23), all of which, of course, have not been included in my estimates of the uncertainty of Eqn(A13).

Finally, if we take $f(N\theta_1)\theta_1^2 \simeq (\pi/N)^2/1.5$--the value which results from averaging over all initial angles θ_1, and using $V(\theta) = (m_a^2/N)\sin(N\theta)$, we obtain

$$(\Omega_a h^2/T_{2.7}^3) = 1.8\times 10^{-1\pm 0.4}(N/6)^{-1.175}(f_a/10^{12}GeV)^{1.175}\Lambda_{200}^{-0.7}\gamma^{-1},$$

The values of f_a and m_a corresponding to the axion-dominated, $\Omega_a=1$ Universe are

$$f_a = 4.3\times 10^{12\pm 0.34}GeV(N/6)(h^2\gamma/T_{2.7}^3)^{0.85}\Lambda_{200}^{0.6}, \tag{A14}$$

$$m_a = 8.3\times 10^{-6\pm 0.34}eV(h^2\gamma/T_{2.7}^3)^{-0.85}\Lambda_{200}^{-0.6}. \tag{A15}$$

The age of the Universe ($\gtrsim$10 Byr) and $h \geq 0.4$ imply that $(\Omega_a h^2/T_{2.7}^3)$ must be less than about 1 which leads to the bound:

$$(f_a/N) \lesssim 0.7\times 10^{12\pm 0.34}GeV\gamma^{0.85}\Lambda_{200}^{0.6}. \tag{A16}$$

Again, let me emphasize that Eqns(A14-16) were derived assuming that $f(N\theta_1)\theta_1^2 = (\pi/N)^2/1.5$, which is an unfounded assumption in an axion-dominated, inflationary Universe.

Figure Captions

Figure 1

Distribution of initial misalignment angles, θ_1, in a Universe which inflates after or during PQ symmetry breaking. Each circle denotes a bubble or fluctuation region. The values of θ_1 were selected at random from the interval $[0, \pi/6]$ --corresponding to N=6; the RMS value of θ_1 for the sample is 0.306 (compared to the expected RMS value of 0.3023). In such a Universe the RMS value of θ_1 has little relevance for us, as we reside within a single bubble or fluctuation region.

Figure 2

Schematic view (edge on) of the three components of our Galaxy, and our position in the Galaxy.

Figure A1

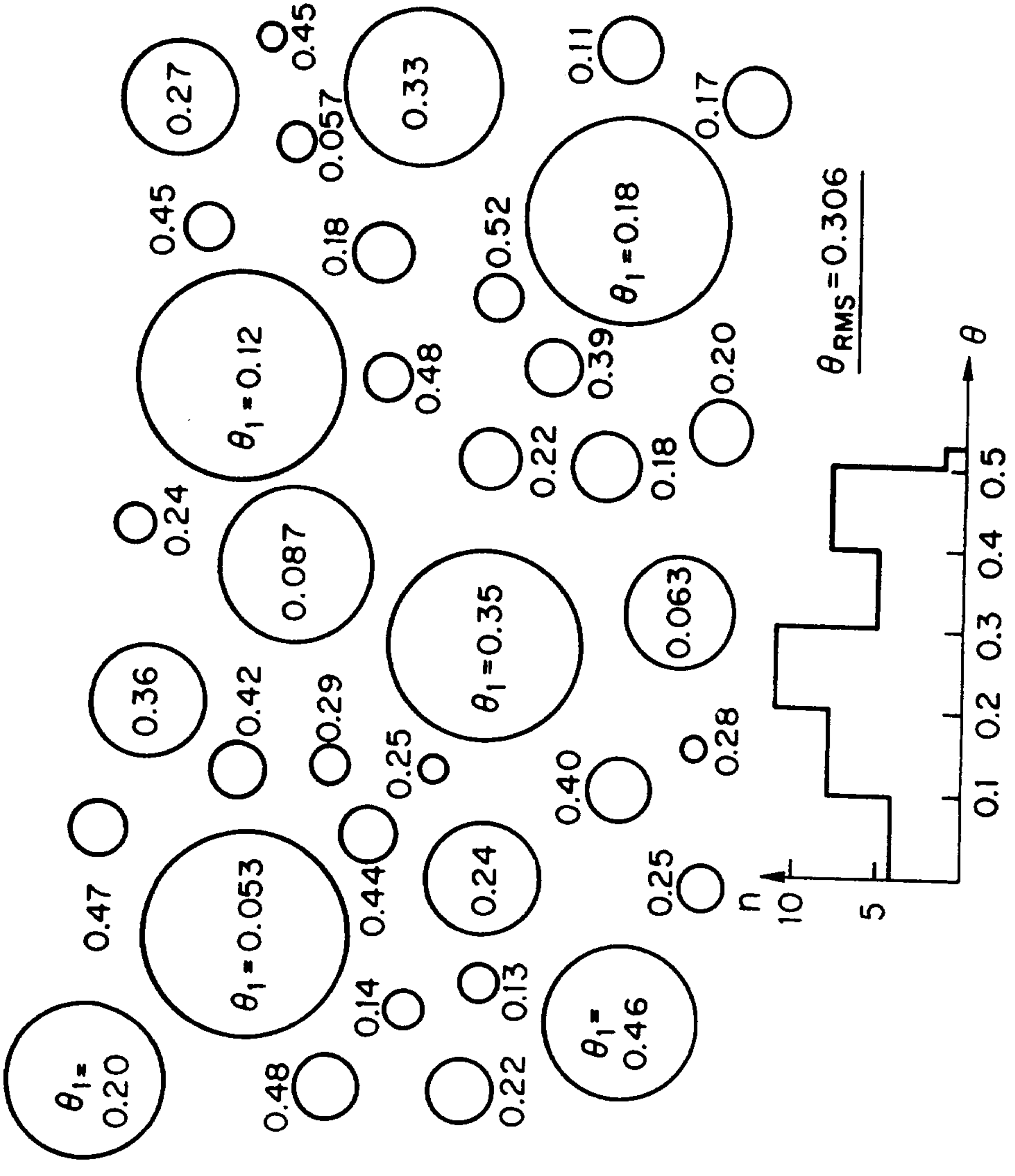
The correction factor cf, as a function of m, where m parameterizes the temperature dependence of the axion mass: $m_a(T) \propto T^{-m}$. Error bars denote the accuracy of the numerical integration. The line cf=0.44+0.25m is a very good fit to the numerical results.

Figure A2

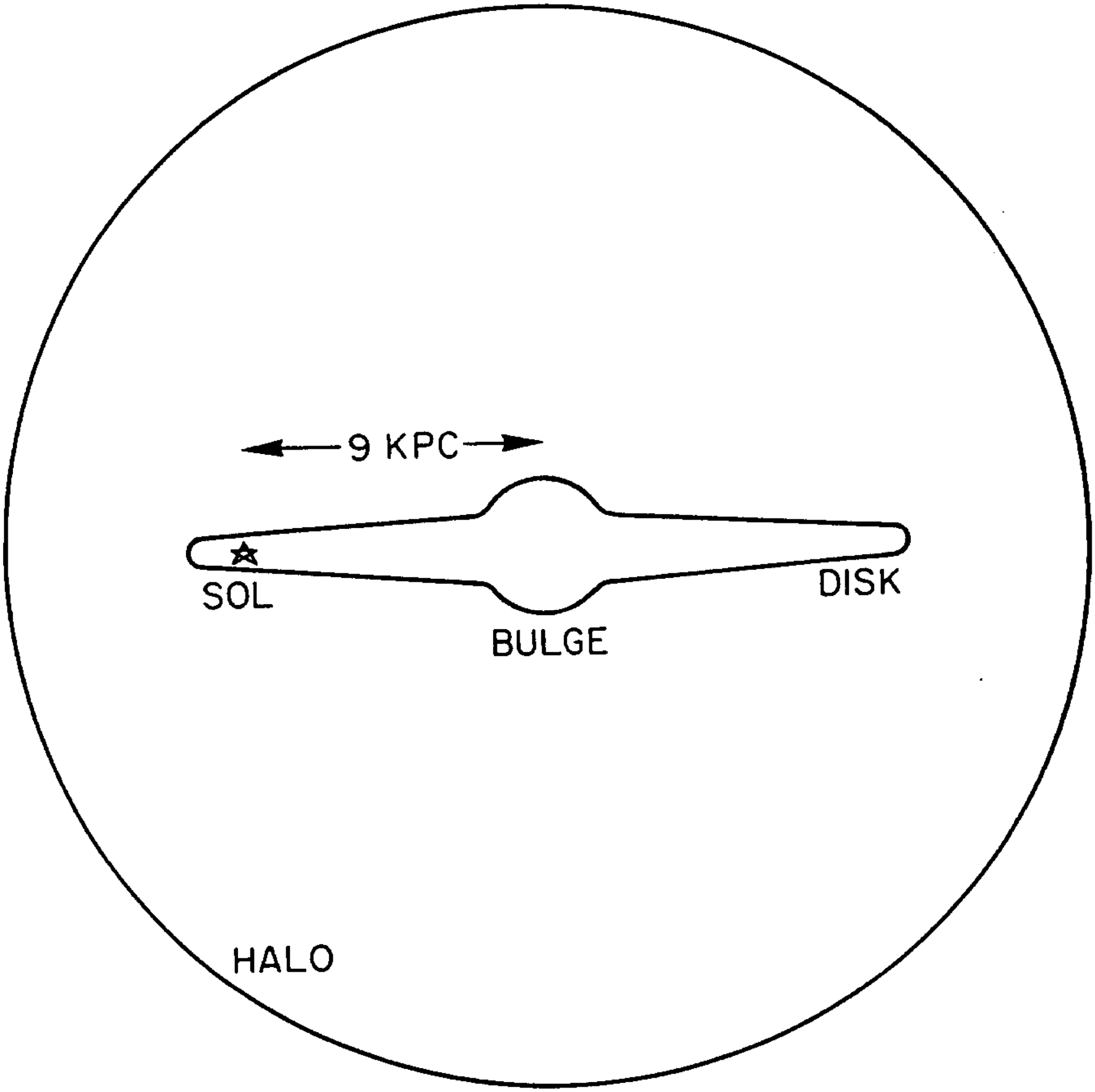
The correction factor due to anharmonic effects, $f(N\theta)$, as a function of $N\theta$, for m=0. 4, and 8. This correction factor is just the ratio of the axion energy density obtained by using $V'(\theta)=(m_a^2/N)\sin(N\theta)$, to that obtained using the linearized form $V'(\theta)=m_a^2\theta$. As expected, for small $N\theta$, $f\simeq 1$. For $N\theta\rightarrow\pi$, $f(N\theta)\rightarrow\infty$; this occurs because for $N\theta=\pi$ the axion field just sits at the maximum of the potential (where $V'=0$).

Figure A3

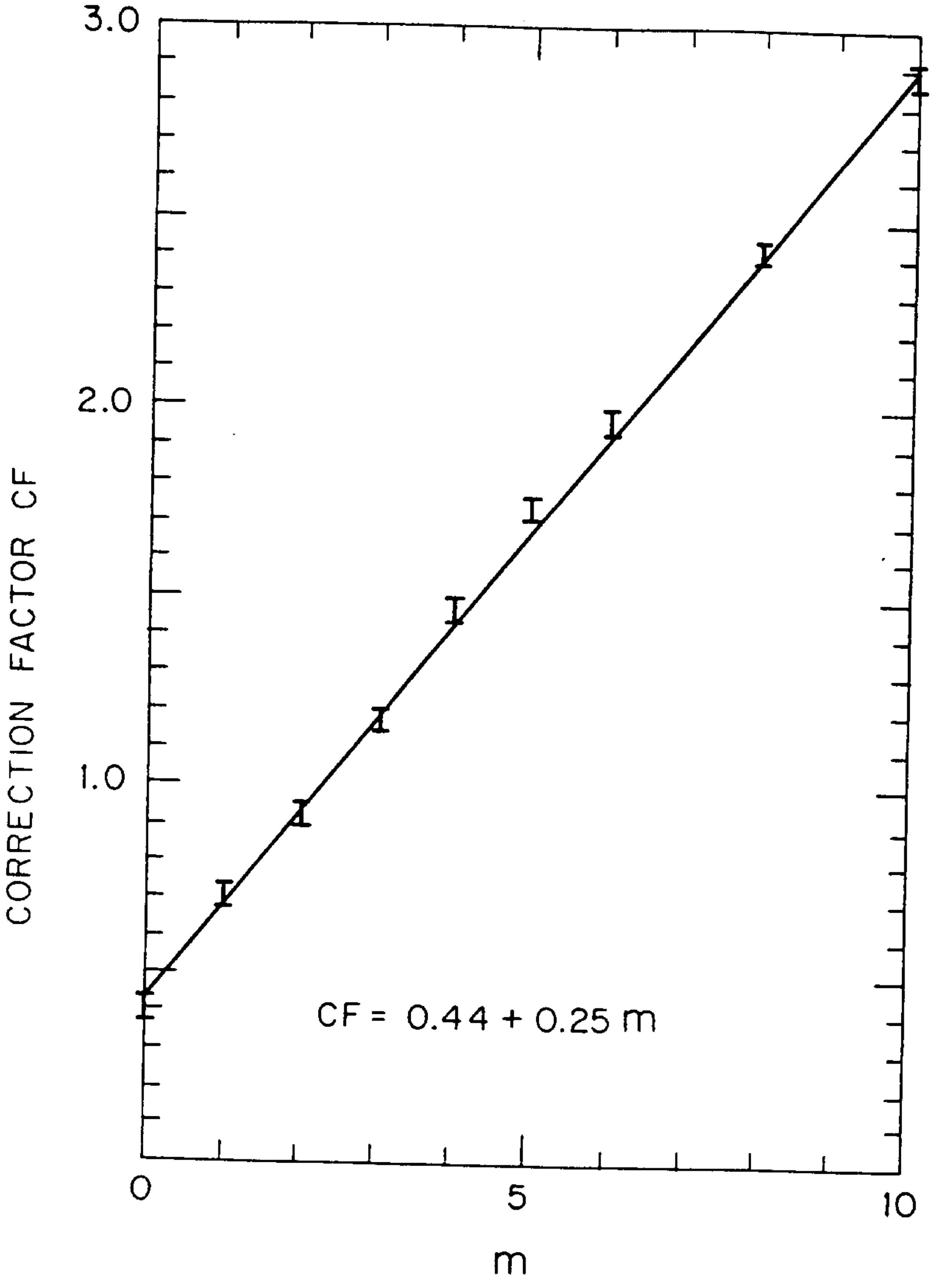
Finite-temperature axion mass divided by the zero-temperature mass, $m_a(T)/m_a$, calculated in the dilute instanton gas approximation for N_f=2, 3, 4, 5 light quark flavors. The calculation is only valid in the high temperature limit $T>>\Lambda$, and due to the $[ln(eT/\Lambda)]^d$ factor in $m_a(T)$, $m_a(T)$ actually vanishes for $T=\Lambda/e\simeq 0.37\Lambda$. The power laws, $0.02(\Lambda/T)^{3.6}$ and $0.3(\Lambda/T)^{3.8}$, nicely bracket the spread in $m_a(T)/m_a$ for N_f=2–5.


$\theta_1 = 0.20$
$\theta_1 = 0.053$
$\theta_1 = 0.46$
$\theta_1 = 0.12$
$\theta_1 = 0.35$
$\theta_1 = 0.18$
0.47
0.48
0.14
0.22
0.13
0.44
0.24
0.25
0.36
0.42
0.29
0.25
0.40
0.28
0.087
0.24
0.063
0.22
0.18
0.20
0.39
0.48
0.52
0.45
0.18
0.27
0.057
0.45
0.33
0.11
0.17
$\theta_{RMS} = 0.306$
n
10
5
0.1
0.2
0.3
0.4
0.5
θ

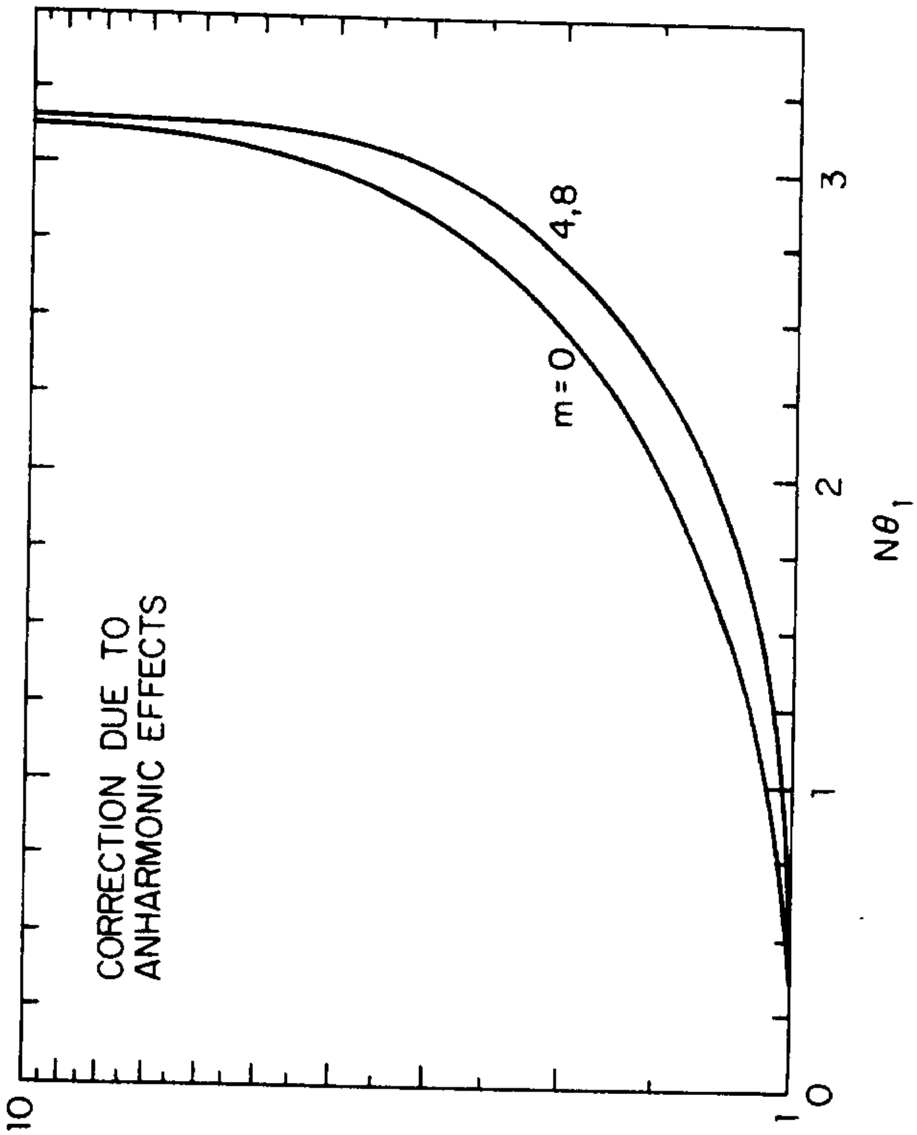
- FIGURE 1 -



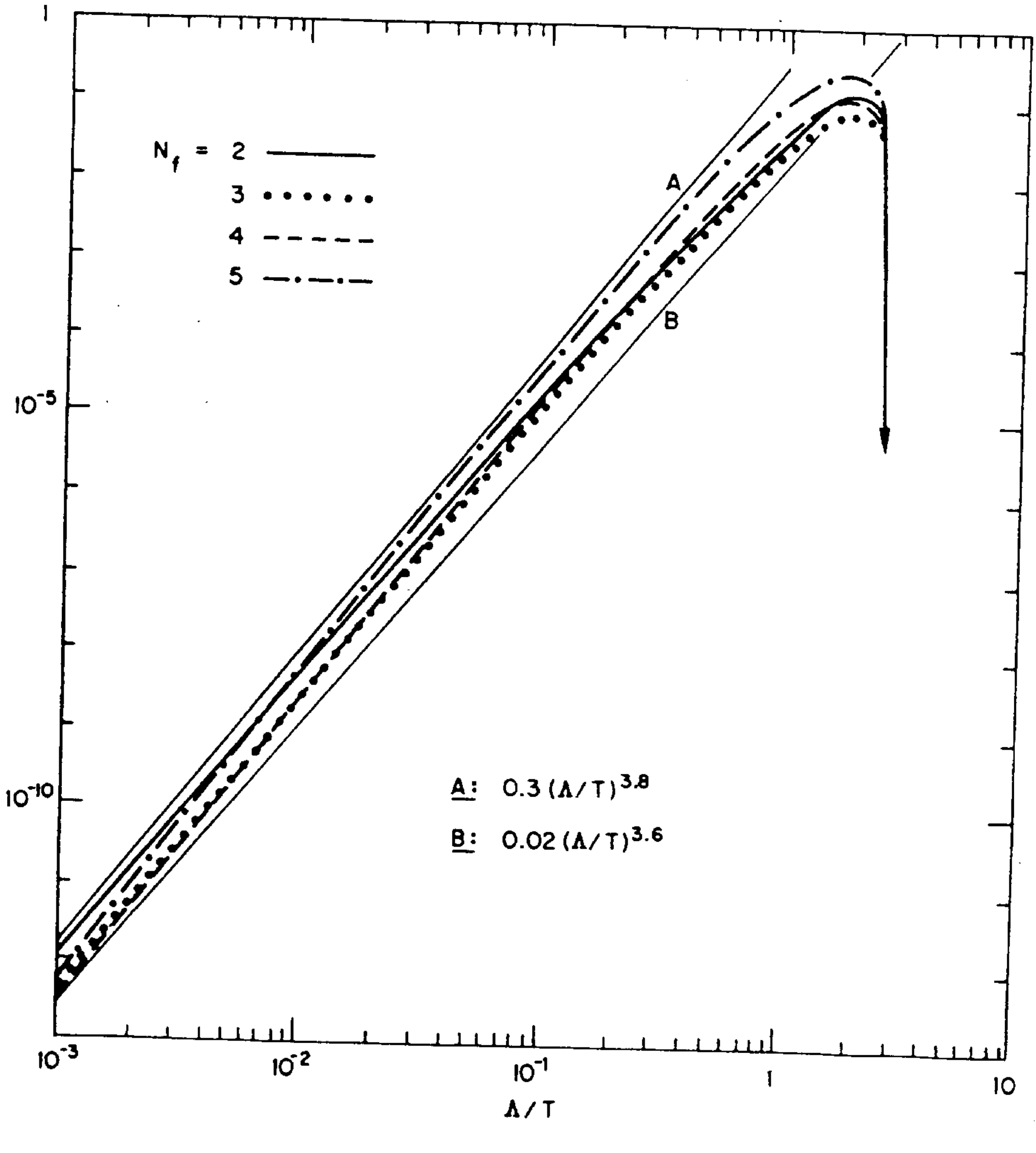
- FIGURE 2 -



-FIGURE A1-


CORRECTION DUE TO
ANHARMONIC EFFECTS
m=0
4,8
CORRECTION FACTOR f(Nθ₁)
Nθ₁
10
1
0
1
2
3

-FIGURE A2-


1
10^{-5}
10^{-10}
N_f = 2
3
4
5
A
B
A: $0.3(\Lambda/T)^{3.8}$
B: $0.02(\Lambda/T)^{3.6}$
10^{-3}
10^{-2}
10^{-1}
1
10
Λ/T

-FIGURE A3-